Today: Lagrange Multipliers

method — used to find critical pts (e.g. max, min, saddles) of functions restricted to curves, surfaces, etc.

e.g. used when restricting to the boundary of a region.

→ SAGEMATH.

What we saw: The places on $x^2+y^2=9$ where $B(x,y)$ is max & min (or any critical point) is where the contours of B would be tangent to $x^2+y^2=9$.

General question: Find critical points of $f(x,y,\ldots)$ restricted to $g(x,y,\ldots)=C$ ← curve or surface or...

$\iff$ Find where contours of f would be tangent to $g(x, y, \dots) = c$.

$\iff$ Find where ∇f & ∇g are parallel.

$$\iff \quad \nabla f = \underset{\substack{\uparrow \\ \text{constant.}}}{\lambda} \nabla g$$

<u>Summary</u>: To find cr pts of $f(x, y, \dots)$ on the set $g(x, y, \dots) = c$,

Solve : $\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$.

Solve for $x, y, (\lambda)$

In our last example, we looked at

$B(x, y) = xy + 2x$

restricted to $g(x, y) = 9$, $g(x, y) = x^2 + y^2$.

<u>To solve:</u>

$$\nabla B = \lambda \nabla g$$

$$(y+2, x) = \lambda (2x, 2y)$$

$g = c \qquad x^2 + y^2 = 9$

$y+2 = \lambda 2x \Rightarrow y^2+2y = \lambda 2xy$

$x = \lambda 2y \Rightarrow x^2 = \lambda 2yx$

$x^2+y^2=9$

$\Rightarrow y^2+2y=x^2$

$x^2+y^2=9$

$\Rightarrow y^2+2y+y^2=9$

$$\boxed{2y^2+2y-9=0}$$

use quadratic formula.

$\rightsquigarrow$ plug in $x=\pm\sqrt{9-y^2}$

... (we would get same 4 pts)

Example. Find the critical points of $f(x,y,z) = 2x-y+z$ on the ellipsoid $x^2+4y^2+z^2=4$

---

let $g(x,y,z) = x^2+4y^2+z^2$

$g(x,y,z)=4$

$\nabla f = \lambda \nabla g \Rightarrow (2,-1,1) = \lambda(2x, 8y, 2z)$

$g=4 \qquad x^2+4y^2+z^2=4.$

$$\left.\begin{aligned} 2 &= \lambda 2x \\ -1 &= \lambda 8y \\ 1 &= \lambda 2z \end{aligned}\right\}$$

solve for $1/\lambda$

$$\frac{1}{\lambda} = x$$

$$\frac{1}{\lambda} = -8y$$

$$\frac{1}{\lambda} = 2z$$

$$x^2 + 4y^2 + z^2 = 4$$

$$x = -8y = 2z$$

$$y = -\frac{x}{8}, \quad z = \frac{x}{2}$$

$$x^2 + \underbrace{\frac{4x^2}{64}}_{\frac{1}{16}} + \frac{x^2}{4} = 4$$

$$16x^2 + x^2 + 4x^2 = 64$$

$$21x^2 = 64$$

$$x^2 = \frac{64}{21}$$

$$x = \pm \frac{8}{\sqrt{21}}$$

$$y = \frac{-x}{8} = \mp \frac{1}{\sqrt{21}}$$

$$z = \frac{x}{2} = \pm \frac{4}{\sqrt{21}}$$

$$\boxed{\text{2 critical pts: } \left(\frac{8}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right), \left(\frac{-8}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right).}$$

$\therefore$ one is max, one is min.

"picture"